Gödel-type Spacetimes in Induced Matter Gravity Theory

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Abstract

A five-dimensional (5D) generalized Gödel-type manifolds are examined in the light of the equivalence problem techniques, as formulated by Cartan. The necessary and sufficient conditions for local homogeneity of these 5D manifolds are derived. The local equivalence of these homogeneous Riemannian manifolds is studied. It is found that they are characterized by three essential parameters k, m^2 and ω: identical triads (k, m^2, ω) correspond to locally equivalent 5D manifolds. An irreducible set of isometrically nonequivalent 5D locally homogeneous Riemannian generalized Gödel-type metrics are exhibited. A classification of these manifolds based on the essential parameters is presented, and the Killing vector fields as well as the corresponding Lie algebra of each class are determined. It is shown that the generalized Gödel-type 5D manifolds admit maximal group of isometry G_r with $r = 7$, $r = 9$ or $r = 15$ depending on the essential parameters k, m^2 and ω. The breakdown of causality in all these classes of homogeneous Gödel-type manifolds are also examined. It is found that in three out of the six irreducible classes the causality can be violated. The unique generalized Gödel-type solution of the induced matter (IM) field equations is found. The question as to whether the induced matter version of general relativity is an effective therapy for these type of causal anomalies of general relativity is also discussed in connection with a recent work by Romero, Tavakol and Zalaletdinov.

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1 Introduction

The field equations of the general relativity theory, which in the usual notation are written in the form

$$G_{\alpha\beta} = \kappa \ T_{\alpha\beta} \ , \qquad (1.1)$$

relate the geometry of the spacetime to its source. The general relativity theory, however, does not prescribe the various forms of matter, and takes over the energy-momentum tensor $T_{\alpha\beta}$ from other branches of physics. In this sense, general relativity (GR) is not a closed theory. The separation between the gravitational field and its source has been often considered as one undesirable feature of GR [1] – [3].

Recently, Wesson and co-workers [4, 5] have introduced a new approach to GR, in which the matter and its role in the determination of the spacetime geometry is given from a purely five-dimensional geometrical point of view. In their five-dimensional (5D) version of general relativity the field equations are given by

$$\widehat{G}_{AB} = 0 \ . \qquad (1.2)$$

Henceforth, the five-dimensional geometrical objects are denoted by overhats and Latin letters are 5D indices and run from 0 to 4. In this new approach to GR the 5D vacuum field equations (1.2) give rise to both curvature and matter in 4D. Indeed, it can be shown [5] that it is always possible to rewrite the fifteen field equations (1.2) as a set of equations such that ten of which are precisely Einstein's field equations (1.1) in 4D with an *induced* energy-momentum

$$
\begin{aligned}
\kappa \ T_{\alpha\beta} \ = \ & \frac{\phi_{\alpha\,;\,\beta}}{\phi} - \frac{\varepsilon}{2\,\phi^2} \left\{ \frac{\phi^* g^*_{\alpha\beta}}{\phi} - g^{**}_{\alpha\beta} + g^{\gamma\delta}\, g^*_{\alpha\gamma}\, g^*_{\beta\delta} - \frac{g^{\gamma\delta}\, g^*_{\gamma\delta}\, g^*_{\alpha\beta}}{2} \right. \\
& \left. + \ \frac{g_{\alpha\beta}}{4} \left[g^{*\gamma\delta}\, g^*_{\gamma\delta} + (g^{\gamma\delta}\, g^*_{\gamma\delta})^2 \right] \right\} \ , \qquad (1.3)
\end{aligned}
$$

where the Greek letters denote 4D indices and run from 0 to 3, $g_{44} \equiv \varepsilon \phi^2$ with $\varepsilon = \pm 1$, $\phi_\alpha \equiv \partial\phi/\partial x^\alpha$, a star denotes $\partial/\partial x^4$, and a semicolon denotes the usual 4D covariant derivative. Obviously, the remaining five equations (a wave equation and four conservation laws) are automatically satisfied by any solution of the 5D vacuum equations (1.2). Thus, not only the matter but also its role in the determination of the geometry of the 4D spacetime can be considered to have a five-dimensional geometrical origin. This approach unifies the gravitational field with its source (not just with a particular field) within a purely 5D geometrical framework. This 5D version of general relativity is often referred to as induced matter gravity theory (IM gravity theory, for short). The IM theory has become a focus of a recent research field [6]. The basic features of the theory have been explored by Wesson and others [7] – [11], whereas the implications for cosmology and astrophysics have been investigated by a number of researchers [12] – [32]. For a fairly updated list of references on IM gravity theory and related issues we refer the reader to Overduin and Wesson [6].

In general relativity, the causal structure of 4D spacetime has locally the same qualitative nature as the flat spacetime of special relativity — causality holds locally. The global question, however, is left open and significant differences can occur. On large scale, the violation of causality is not excluded. Actually, it has long been known that there are solutions to the Einstein field equations which possess causal anomalies in the form of closed timelike curves. The famous solution found by Gödel [33] in 1949 might not be

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the first but it certainly is the best known example of a cosmological model which makes it apparent that general relativity, as it is normally formulated, does not exclude the existence of closed timelike world lines, despite its Lorentzian character which leads to the local validity of the causality principle. Owing to its striking properties Gödel's model has a well-recognized importance and has to a certain extent motivated the investigations on rotating cosmological Gödel-type models and on causal anomalies in the framework of general relativity [34] – [52] and other theories of gravitation [53] – [63].

Two recent articles have been concerned with *five-dimensional* Gödel-type spacetimes. Firstly in Ref. [64] the main geometrical properties of five-dimensional Riemannian manifolds endowed with a 5D counterpart of the 4D Gödel-type metric of general relativity were investigated. Among several results, an irreducible set of isometrically nonequivalent 5D homogeneous (locally) Gödel-type metrics were exhibited. Therein it was also shown that, apart from the degenerated Gödel-type metric, in all classes of homogeneous Gödel-type geometries there is breakdown of causality. As no use of any particular field equations was made in this first paper, its results hold for any 5D Gödel-type manifolds regardless of the underlying 5D Kaluza-Klein gravity theory. In the second article [65] the classes of 5D Gödel-type spacetimes discussed in [64] were investigated from a more physical viewpoint. Particularly, it was examined the question as to whether the induced matter theory of gravitation permits the family of noncausal solutions of Gödel-type metrics studied in [64]. It was shown that the IM gravity excludes this class of 5D Gödel-type non-causal geometries as solution to its field equations.

In both articles [64, 65] the 5D Gödel-type family of metrics discussed is the simplest 5D class of geometries for which the section $u = $ const (u is the extra coordinate) is the 4D Gödel-type metric of general relativity. Actually the 5D Gödel-type line element of both papers does not depend on the fifth coordinate u, and therefore as regards to the IM theory a radiation-like equation of state is an underlying assumption of both articles. However, it is well know [6] that the dependence of the 5D metric on the extra coordinate is necessary to ensure that the 5D IM theory permits the induction of matter of a very general type in 4D.

In this work, on the one hand, we shall examine the main geometrical properties of a class of *generalized* Gödel-type geometries in which the 5D metric depends on the fifth coordinate, generalizing therefore the results found in Ref. [64]. On the other hand, we shall also investigate the question as to whether the induced matter gravity theory, as formulated by Wesson and co-workers [4, 5], admits these generalized Gödel-type metrics as solutions to its field equations, thus also extending the investigations of Ref. [65].

The outline of this article is as follows. In the next section we present a summary of some important prerequisites for Section 3, where using the equivalence problem techniques as formulated by Cartan [66] we derive the necessary and sufficient conditions for local homogeneity of this class of 5D generalized Gödel-type manifolds. In Section 3 we also exhibit an irreducible set of isometrically nonequivalent homogeneous generalized Gödel-type metrics. In Section 4 we discuss the integration of the Killing equations and present the Killing vector fields as well as the corresponding Lie algebra for all homogeneous generalized Gödel-type metrics. In the last section we examine whether the IM field equations permit solutions of this generalized Gödel-type class of geometries. The unique solution of this type is found therein. The question as to whether the IM version of general relativity rules out the existence of closed timelike curves of Gödel type is also discussed (Section 5) in connection with a recent paper by Romero *et al.* [67].

2 Prerequisites

The arbitrariness in the choice of coordinates in the metric theories of gravitation gives rise to the problem of deciding whether or not two manifolds whose metrics g and $\tilde{g}$ are given explicitly in terms of coordinates, viz.,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \text{and} \qquad d\tilde{s}^2 = \tilde{g}_{\mu\nu}\, d\tilde{x}^\mu\, d\tilde{x}^\nu \,, \tag{2.1}$$

are locally isometric. This is the so-called equivalence problem (see Cartan [66] for the local equivalence of n-dimensional Riemannian manifolds, Karlhede [68] and MacCallum [69] for the special case $n = 4$ of general relativity).

The Cartan solution [66] to the equivalence problem for Riemannian manifolds can be summarized as follows. Two n-dimensional Lorentzian Riemannian manifolds $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$ are locally equivalent if there exist coordinate and generalized n-dimensional Lorentz transformations such that the following *algebraic* equations relating the frame components of the curvature tensor and their covariant derivatives:

$$\begin{aligned}
R^A{}_{BCD} &= \tilde{R}^A{}_{BCD}\,, \\
R^A{}_{BCD;M_1} &= \tilde{R}^A{}_{BCD;M_1}\,, \\
R^A{}_{BCD;M_1M_2} &= \tilde{R}^A{}_{BCD;M_1M_2}\,, \\
&\vdots \\
R^A{}_{BCD;M_1...M_{p+1}} &= \tilde{R}^A{}_{BCD;M_1...M_{p+1}}
\end{aligned} \tag{2.2}$$

are compatible as *algebraic* equations in $\left(x^\mu, \xi^A\right)$. Here and in what follows we use a semicolon to denote covariant derivatives. Note that x^μ are coordinates on the manifold $\mathcal{M}_n$ while ξ^A parametrize the group of allowed frame transformations [n-dimensional generalized Lorentz group usually denoted [70] by $O(n-1,1)$]. Reciprocally, equations (2.2) imply local equivalence between the n-dimensional manifolds $\mathcal{M}_n$ and $\widetilde{\mathcal{M}}_n$.

In practice, a fixed frame is chosen to perform the calculations so that only coordinates appear in the components of the curvature tensor, i.e. there is no explicit dependence on the parameters ξ^A of the generalized Lorentz group.

Another important practical point to be considered, once one wishes to test the local equivalence of two Riemannian manifolds, is that before attempting to solve eqs. (2.2) one can extract and compare partial pieces of information at each step of differentiation as, for example, the number $\{t_0, t_1, \ldots, t_p\}$ of functionally independent functions of the coordinates x^μ contained in the corresponding set

$$I_p = \{R^A{}_{BCD}\,, R^A{}_{BCD;M_1}\,, R^A{}_{BCD;M_1M_2}\,, \ldots, R^A{}_{BCD;M_1M_2...M_p}\}\,, \tag{2.3}$$

and the isotropy subgroup $\{H_0, H_1, \ldots, H_p\}$ of the symmetry group G_r under which the set corresponding I_p is invariant. They must be the same for each step $q = 0, 1, \cdots, p$ if the manifolds are locally equivalent.

In practice it is also important to note that in calculating the curvature and its covariant derivatives, in a chosen frame, one can stop as soon as one reaches a step at which the pth derivatives (say) are algebraically expressible in terms of the previous ones, and the residual isotropy group (residual frame freedom) at that step is the same isotropy group of the previous step, i.e. $H_p = H_{(p-1)}$. In this case further differentiation will not yield

any new piece of information. Actually, if $H_p = H_{(p-1)}$ and, in a given frame, the pth derivative is expressible in terms of its predecessors, for any $q > p$ the qth derivatives can all be expressed in terms of the 0th, 1st, $\cdots$, $(p-1)^{th}$ derivatives [66, 69].

Since there are t_p essential coordinates, in 5D clearly $5 - t_p$ are ignorable, so the isotropy group will have dimension $s = \dim(H_p)$, and the group of isometries of the metric will have dimension r given by (see Cartan [66])

$$r = s + 5 - t_p ,\tag{2.4}$$

acting on an orbit with dimension

$$d = r - s = 5 - t_p .\tag{2.5}$$

3   Homogeneity and Nonequivalent Metrics

The line element of the five-dimensional *generalized* Gödel-type manifolds $\mathcal{M}_5$ we are concerned with is given by

$$d\hat{s}^2 = dt^2 + 2\,H(x)\,dt\,dy - dx^2 - G(x)\,dy^2 - \widetilde{F}^2(\tilde{u})\,(d\tilde{z}^2 + d\tilde{u}^2) ,\tag{3.1}$$

where $H(x)$, $G(x)$ and $\widetilde{F}(\tilde{u})$ are arbitrary real functions. By a suitable choice of coordinates the line element (3.1) can be brought into the form

$$d\hat{s}^2 = [\,dt + H(x)\,dy\,]^2 - dx^2 - D^2(x)\,dy^2 - F^2(u)\,dz^2 - du^2 ,\tag{3.2}$$

where $D^2(x) = G + H^2$ and u clearly is a new fifth coordinate.

At an arbitrary point of $\mathcal{M}_5$ one can choose the following set of linearly independent one-forms $\widehat{\Theta}^A$:

$$\widehat{\Theta}^0 = dt + H(x)\,dy , \quad \widehat{\Theta}^1 = dx , \quad \widehat{\Theta}^2 = D(x)\,dy , \quad \widehat{\Theta}^3 = F(u)\,dz , \quad \widehat{\Theta}^4 = du , \quad (3.3)$$

such that the Gödel-type line element (3.2) can be written as

$$d\hat{s}^2 = \widehat{\eta}_{AB}\,\widehat{\Theta}^A\,\widehat{\Theta}^B = (\widehat{\Theta}^0)^2 - (\widehat{\Theta}^1)^2 - (\widehat{\Theta}^2)^2 - (\widehat{\Theta}^3)^2 - (\widehat{\Theta}^4)^2 .\tag{3.4}$$

Here and in what follows capital letters are 5D Lorentz frame indices and run from 0 to 4; they are raised and lowered with Lorentz matrices $\widehat{\eta}^{AB} = \widehat{\eta}_{AB} = \mathrm{diag}(+1, -1, -1, -1, -1)$, respectively.

Using as input the one-forms (3.3) and the Lorentz frame (3.4), the computer algebra package CLASSI [69, 71], e.g., gives the following nonvanishing Lorentz frame components $\widehat{R}_{ABCD}$ of the curvature:

$$\widehat{R}_{0101} = \widehat{R}_{0202} = -\frac{1}{4}\left(\frac{H'}{D}\right)^2 ,\tag{3.5}$$

$$\widehat{R}_{0112} = \frac{1}{2}\left(\frac{H'}{D}\right)' ,\tag{3.6}$$

$$\widehat{R}_{1212} = \frac{D''}{D} - \frac{3}{4}\left(\frac{H'}{D}\right)^2 ,\tag{3.7}$$

$$\widehat{R}_{3434} = \frac{\ddot{F}}{F} ,\tag{3.8}$$

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where the prime and the dot denote, respectively, derivative with respect to x and u.

For 5D (local) homogeneity from eq. (2.5) one must have $t_q = 0$ for $q = 0, 1, \cdots p$, that is, the number of functionally independent functions of the coordinates x^μ in the set I_p must be zero. Therefore, from eqs. (3.5) – (3.8) we conclude that for 5D homogeneity it is necessary that

$$\frac{H'}{D} = \text{const} \equiv -2\,\omega\,, \tag{3.9}$$

$$\frac{D''}{D} = \text{const} \equiv m^2\,, \tag{3.10}$$

$$\frac{\ddot{F}}{F} = \text{const} \equiv k\,. \tag{3.11}$$

The above necessary conditions are also sufficient for 5D local homogeneity. Indeed, under these conditions the nonvanishing frame components of the curvature reduce to

$$\widehat{R}_{0101} = \widehat{R}_{0202} = -\omega^2\,, \tag{3.12}$$
$$\widehat{R}_{1212} = m^2 - 3\,\omega^2\,, \tag{3.13}$$
$$\widehat{R}_{3434} = k\,. \tag{3.14}$$

Following Cartan's method for the local equivalence, we calculate the first covariant derivative of the Riemann tensor. One obtains the following non-null covariant derivatives of the curvature:

$$\widehat{R}_{0112;1} = \widehat{R}_{0212;2} = \omega\,(m^2 - 4\,\omega^2)\,. \tag{3.15}$$

Clearly, regardless of the value of the constant k, the first covariant derivative of the curvature is algebraically expressible in terms of the Riemann tensor. Moreover, the number of functionally independent functions of the coordinates x^μ among the components of the curvature and its first covariant derivative is zero ($t_0 = t_1 = 0$). As far as the dimension of the residual isotropy group is concerned we distinguish three different classes of locally homogeneous 5D generalized Gödel-type curved manifolds, according to the relevant parameters m^2, ω and k, namely [72]

1. $\dim(H_0) = \dim(H_1) = 2$ when

    a) $\omega \neq 0$, any real k, $m^2 \neq 4\,\omega^2$;

    b) $\omega = 0$, $k \neq 0$, $m^2 \neq 0$;

2. $\dim(H_0) = \dim(H_1) = 4$ when

    a) $\omega \neq 0$, any real k, $m^2 = 4\,\omega^2$;

    b) $\omega = 0$, $k = 0$, $m^2 \neq 0$;

    c) $\omega = 0$, $k \neq 0$, $m^2 = 0$;

3. $\dim(H_0) = \dim(H_1) = 10$ when $\omega = k = m^2 = 0$.

Thus, from eqs. (2.4) and (2.5) one finds that the locally homogeneous 5D generalized Gödel-type manifolds admit a (local) G_r, with either $r = 7$, $r = 9$, or $r = 15$ acting on an orbit of dimension $d = 5$, that is on the manifold $\mathcal{M}_5$.

The above results can be collected together in the following theorems:

Theorem 1 *The necessary and sufficient conditions for a five-dimensional generalized Gödel-type manifold to be locally homogeneous are those given by equations (3.9) – (3.11).*

Theorem 2 *The five-dimensional homogeneous generalized Gödel-type manifolds are locally characterized by three independent real parameters ω, k and m^2: identical triads (ω, k, m^2) specify locally equivalent manifolds.*

Theorem 3 *The five-dimensional locally homogeneous generalized Gödel-type manifolds admit group of isometry G_r with*

  (i) $r = 7$ if either of the above conditions (1.a) and (1.b) is fulfilled;

  (ii) $r = 9$ if one of the above set of conditions (2.a), (2.b) and (2.c) is fulfilled;

  (iii) $r = 15$ if the above condition (3) is satisfied.

We shall now focus our attention on the irreducible set of isometrically nonequivalent homogeneous generalized Gödel-type metrics. These nonequivalent classes of metrics can be obtained by a similar procedure to that used by Rebouças and Tiomno [41], namely by integrating equations (3.9) – (3.11), and eliminating through coordinate transformations the non-essential integration constants taking into account the relevant parameters according to the above theorem 2. For the sake of brevity, however, we shall only present the irreducible classes without going into details of calculations. It turns out that one ought to distinguish six classes of metrics according to:

Class I : $m^2 > 0$, any real k, $\omega \neq 0$. The line element for this class of homogeneous generalized Gödel-type manifolds can always be brought [in cylindrical coordinates (r, ϕ, z)] into the form

$$d\hat{s}^2 = [\,dt + H(r)\,d\phi\,]^2 - D^2(r)\,d\phi^2 - dr^2 - F^2(u)\,dz^2 - du^2 \tag{3.16}$$

with the metric functions given by

$$H(r) \;\; = \;\; \frac{2\,\omega}{m^2}\,[1 - \cosh{(mr)}]\,, \tag{3.17}$$

$$D(r) \;\; = \;\; m^{-1}\,\sinh{(mr)}\,, \tag{3.18}$$

$$F(u) = \begin{cases} \alpha^{-1}\,\sin{(\alpha\,u)} & \text{if} & k = -\alpha^2 < 0\,, \\ u & \text{if} & k = 0\,, \\ \alpha^{-1}\,\sinh{(\alpha\,u)} & \text{if} & k = \alpha^2 > 0\,. \end{cases} \tag{3.19}$$

According to theorem 3 the possible isometry groups for this class are either G_7 (for $m^2 \neq 4\,\omega^2$) or G_9 (when $m^2 = 4\,\omega^2$), irrespective of the value of k.

Class II : $m^2 = 0$, any real k, $\omega \neq 0$. The line element for this class can be brought into the form (3.16), with the metric function $F(u)$ given by (3.19), but now the functions $H(r)$ and $D(r)$ are given by

$$H(r) = -\,\omega\,r^2 \qquad \text{and} \qquad D(r) = r\,. \tag{3.20}$$

For this class from theorem 3 there is a group G_7 of isometries, regardless of the value of k.

Class III : $m^2 \equiv -\mu^2 < 0$, any real k, $\omega \neq 0$. Similarly for this class the line element reduces to (3.16) with $F(u)$ given by (3.19) and

$$H(r) = \frac{2\,\omega}{\mu^2}\,[\cos(\mu r) - 1]\,, \tag{3.21}$$

$$D(r) = \mu^{-1}\,\sin(\mu r)\,. \tag{3.22}$$

From theorem 3, regardless the value of k for this class there is a group G_7 of isometries.

Class IV : $m^2 \neq 0$, any real k, and $\omega = 0$. We shall refer to this class as degenerated Gödel-type manifolds, since the cross term in the line element, related to the rotation ω in 4D Gödel model, vanishes. By a trivial coordinate transformation one can make $H = 0$ with $D(r)$ given, respectively, by (3.18) or (3.22) depending on whether $m^2 > 0$ or $m^2 \equiv -\mu^2 < 0$. The function $F(u)$ depends on the sign of k and is again given by (3.19). For this class according to theorem 3 one may have either a G_7 for $k \neq 0$, or a G_9 for $k = 0$.

Class V : $m^2 = 0$, $k \neq 0$, and $\omega = 0$. By a trivial coordinate transformation one can make $H = 0$, $D = r$ and $F(u) = \alpha^{-1}\sin(\alpha\,u)$ or $F(u) = \alpha^{-1}\sinh(\alpha\,u)$ depending on whether $k < 0$ or $k > 0$, respectively. From theorem 3 there is a group G_9 of isometries.

Class VI : $m^2 = 0$, $k = 0$, and $\omega = 0$. From (3.12) – (3.14) this corresponds to the 5D flat manifold. Therefore, one can make $H = 0$, $D(r) = r$ and $F(u) = u$. Theorem 3 ensures that there is a group G_{15} of isometries.

4 Killing Vector Fields

In this section we shall present the infinitesimal generators of isometries of the 5D homogeneous generalized Gödel-type manifolds, whose line element (3.16) can be brought into the Lorentzian form (3.4) with $\widehat{\Theta}^A$ given by

$$\widehat{\Theta}^0 = dt + H(r)\,d\phi\,, \quad \widehat{\Theta}^1 = dr\,, \quad \widehat{\Theta}^2 = D(r)\,d\phi\,, \quad \widehat{\Theta}^3 = F(u)\,dz\,, \quad \widehat{\Theta}^4 = du\,, \quad (4.1)$$

where the functions $H(r)$, $D(r)$ and $F(u)$ depend upon the essential parameters m^2, k and ω according to the above classes of locally homogeneous manifolds.

Denoting the coordinate components of a generic Killing vector field $\widehat{K}$ by $\widehat{K}^u \equiv (Q, R, S, \bar{Z}, U)$, where $Q, R, S, \bar{Z}$ and U are functions of all coordinates t, r, ϕ, z, u, then the fifteen Killing equations

$$\widehat{K}_{(A;B)} \equiv \widehat{K}_{A;B} + \widehat{K}_{B;A} = 0 \tag{4.2}$$

can be written in the Lorentz frame (3.4) – (4.1) as

$$T_t = 0\,, \qquad T_u - U_t = 0\,, \tag{4.3}$$

$$R_r = 0\,, \qquad U_r + R_u = 0\,, \tag{4.4}$$

$$U_u = 0\,, \tag{4.5}$$

$$D\,(T_r - R_t) - H_r P = 0\,, \tag{4.6}$$

$$DP_u + U_\phi - HU_t = 0\,, \tag{4.7}$$

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$$T_\phi + H_r R - D P_t = 0 \,, \tag{4.8}$$

$$R_\phi - H R_t - D_r P + D P_r = 0 \,, \tag{4.9}$$

$$P_\phi - H P_t + D_r R = 0 \,, \tag{4.10}$$

$$T_z - F Z_t = 0 \,, \tag{4.11}$$

$$F Z_r + R_z = 0 \,, \tag{4.12}$$

$$Z_z + U F_u = 0 \,, \tag{4.13}$$

$$U_z + F Z_u - Z F_u = 0 \,, \tag{4.14}$$

$$D P_z + F \left(Z_\phi - H Z_t \right) = 0 \,, \tag{4.15}$$

where the subscripts denote partial derivatives, and where we have made

$$T \equiv H S + Q, \qquad P \equiv D S, \qquad \text{and} \qquad Z \equiv F \bar{Z} \tag{4.16}$$

to make easier the comparison and the use of the results obtained in [42]. To this end we note that with the changes $u \to z$ and $U \to Z$ the above equations (4.3) – (4.10) are formally identical to the Killing equations (4) to (11) of [42]. However, in the equations (4.3) – (4.10) the functions T, R, P, U depend additionally on the fifth coordinate u. Taking into account this similitude, the integration of the Killing equations (4.3) – (4.15) can be obtained in two steps as follows. First, by analogy with (4) to (11) of Ref. [42] one integrates (4.3) – (4.10), but at this step instead of the integration constants one has integration functions of the fifth coordinate u. Second, one uses the remaining eqs. (4.11) – (4.15) to achieve explicit forms for these integration functions and to obtain the last component U of the generic Killing vector K.

We have used the above two-steps procedure to integrate the Killing equations (4.3) – (4.15) for all class of homogeneous generalized Gödel-type manifolds. However, for the sake of brevity, we shall only present the Killing vector fields and the corresponding Lie algebras without going into details of calculations, which can be verified by using, for example, the computer algebra program KILLNF, written in CLASSI by Åman [71].

Class I : $m^2 > 0$, any real k, $\omega \neq 0$. In the integration of the Killing equation for this general class one is led to distinguish two different subclasses of solutions depending on whether $m^2 \neq 4\,\omega^2$ or $m^2 = 4\,\omega^2$. We shall refer to these subclasses as classes Ia and Ib, respectively.

Class Ia : $m^2 > 0$, any real k, $m^2 \neq 4\,\omega^2$. In the coordinate basis in which as (3.16) is given, a set of linearly independent Killing vector fields K_N (N is an enumerating index) is given by

$$K_1 \; = \; \partial_t \,, \qquad K_2 = \frac{2\,\omega}{m}\,\partial_t - m\,\partial_\phi \,, \tag{4.17}$$

$$K_3 \; = \; -\frac{H}{D}\,\sin\phi\,\partial_t + \cos\phi\,\partial_r - \frac{D_r}{D}\,\sin\phi\,\partial_\phi \,, \tag{4.18}$$

$$K_4 \; = \; -\frac{H}{D}\,\cos\phi\,\partial_t - \sin\phi\,\partial_r - \frac{D_r}{D}\,\cos\phi\,\partial_\phi \,, \tag{4.19}$$

$$K_5 \; = \; \sin z\,\partial_u + \frac{F_u}{F}\,\cos z\,\partial_z \,, \tag{4.20}$$

$$K_6 \; = \; \cos z\,\partial_u - \frac{F_u}{F}\,\sin z\,\partial_z \,, \tag{4.21}$$

$$K_7 \; = \; \partial_z \,. \tag{4.22}$$

The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = -m\,K_4\,, \qquad [K_2, K_4] = m\,K_3\,, \qquad [K_3, K_4] = m\,K_2\,, \qquad (4.23)$$

$$[K_5, K_6] = -\,k\,K_7\,, \qquad [K_5, K_7] = -K_6\,, \qquad [K_6, K_7] = K_5\,. \qquad (4.24)$$

Therefore the corresponding algebra is $\mathcal{L}_{Ia} = \mathcal{L}_k \oplus \tau \oplus so\,(2,1)$. Here and in what follows the symbols $\oplus$ and $\Subset$ denote and direct and semi-direct sum of sub-algebras, and the sub-algebra $\mathcal{L}_k$ is $so\,(3)$ for $k < 0$, $so\,(2,1)$ for $k > 0$, and $t^2 \Subset so\,(2)$ for $k = 0$. For the present class $\mathcal{L}_k$ is generated by K_5, K_6 and K_7, the symbol τ is associated to the time translation K_1, and finally the infinitesimal generators of sub-algebra $so\,(2,1)$ are K_2, K_3 and K_4.

Class Ib : $m^2 = 4\,\omega^2$, any real k, $\omega \neq 0$. For this class the Killing vector fields are

$$K_1 \;=\; \partial_t\,, \qquad K_2 = \partial_t - m\,\partial_\phi\,, \tag{4.25}$$

$$K_3 \;=\; -\frac{H}{D}\,\sin\phi\,\partial_t + \cos\phi\,\partial_r - \frac{D_r}{D}\,\sin\phi\,\partial_\phi\,, \tag{4.26}$$

$$K_4 \;=\; -\frac{H}{D}\,\cos\phi\,\partial_t - \sin\phi\,\partial_r - \frac{D_r}{D}\,\cos\phi\,\partial_\phi\,, \tag{4.27}$$

$$K_5 \;=\; -\frac{H}{D}\,\cos(mt + \phi)\,\partial_t + \sin(mt + \phi)\,\partial_r + \frac{1}{D}\,\cos(mt + \phi)\,\partial_\phi\,, \tag{4.28}$$

$$K_6 \;=\; -\frac{H}{D}\,\sin(mt + \phi)\,\partial_t - \cos(mt + \phi)\,\partial_r + \frac{1}{D}\,\sin(mt + \phi)\,\partial_\phi\,, \tag{4.29}$$

$$K_7 \;=\; \sin z\,\partial_u + \frac{F_u}{F}\,\cos z\,\partial_z\,, \tag{4.30}$$

$$K_8 \;=\; \cos z\,\partial_u - \frac{F_u}{F}\,\sin z\,\partial_z\,, \tag{4.31}$$

$$K_9 \;=\; \partial_z\,, \tag{4.32}$$

whose Lie algebra is given by

$$[K_1, K_5] = -m\,K_6\,, \qquad [K_1, K_6] = m\,K_5\,, \qquad [K_2, K_3] = -m\,K_4\,, \qquad (4.33)$$

$$[K_2, K_4] = m\,K_3\,, \qquad [K_3, K_4] = m\,K_2\,, \qquad [K_5, K_6] = m\,K_1\,, \qquad (4.34)$$

$$[K_7, K_8] = -\,k\,K_9\,, \qquad [K_7, K_9] = -K_8\,, \qquad [K_8, K_9] = K_7\,. \qquad (4.35)$$

So, the corresponding algebra for this case is $\mathcal{L}_{Ib} = \mathcal{L}_k \oplus so\,(2,1) \oplus so\,(2,1)$. As in the previous class the sub-algebra $\mathcal{L}_k$ depends on the sign of k, and here is generated by K_7, K_8 and K_9. The two sub-algebras $so\,(2,1)$ are generated by the Killing vector fields K_1, K_5, K_6 and K_2, K_3, K_4.

Class II : $m^2 = 0$, any real k, $\omega \neq 0$. For this class the Killing vector fields turns out to be the following:

$$K_1 \;=\; \partial_t\,, \qquad K_2 = \partial_\phi\,, \tag{4.36}$$

$$K_3 \;=\; -\,\omega\,r\,\sin\phi\,\partial_t - \cos\phi\,\partial_r + \frac{1}{r}\,\sin\phi\,\partial_\phi\,, \tag{4.37}$$

$$K_4 \;=\; -\,\omega\,r\,\cos\phi\,\partial_t + \sin\phi\,\partial_r + \frac{1}{r}\,\cos\phi\,\partial_\phi\,, \tag{4.38}$$

$$K_5 \;=\; \sin z\,\partial_u + \frac{F_u}{F}\,\cos z\,\partial_z\,, \tag{4.39}$$

$$K_6 \;=\; \cos z\,\partial_u - \frac{F_u}{F}\,\sin z\,\partial_z\,, \tag{4.40}$$

$$K_7 \;=\; \partial_z\,. \tag{4.41}$$

The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = K_4 \,, \quad [K_2, K_4] = -K_3 \,, \quad [K_3, K_4] = 2\,\omega\,K_1 \,, \tag{4.42}$$

$$[K_5, K_6] = -\,k\,K_7 \,, \qquad [K_5, K_7] = -K_6 \,, \qquad [K_6, K_7] = K_5 \,. \tag{4.43}$$

Therefore, the corresponding algebra for this case is $\mathcal{L}_{II} = \mathcal{L}_k \oplus \mathcal{L}_4$. The sub-algebra $\mathcal{L}_4$ is generated by K_1, K_2, K_3 and K_4. This algebra $\mathcal{L}_4$ is soluble and does not contain abelian 3D sub-algebras; it is classified as type III with $q = 0$ by Petrov [73]. The sub-algebra $\mathcal{L}_k$ is the same of the previous classes and is generated by K_5, K_6 and K_7.

Class III : $m^2 \equiv -\mu^2 < 0$, any real k, $\omega \neq 0$. For this class the set of linearly independent Killling vector fields we have found is given by

$$K_1 \;\; = \;\; \partial_t \,, \quad K_2 = \frac{2\,\omega}{\mu}\,\partial_t + \mu\,\partial_\phi \,, \tag{4.44}$$

$$K_3 \;\; = \;\; -\frac{H}{D}\,\sin\phi\,\partial_t + \cos\phi\,\partial_r - \frac{D_r}{D}\,\sin\phi\,\partial_\phi \,, \tag{4.45}$$

$$K_4 \;\; = \;\; -\frac{H}{D}\,\cos\phi\,\partial_t - \sin\phi\,\partial_r - \frac{D_r}{D}\,\cos\phi\,\partial_\phi \,, \tag{4.46}$$

$$K_5 \;\; = \;\; \sin z\,\partial_u + \frac{F_u}{F}\,\cos z\,\partial_z \,, \tag{4.47}$$

$$K_6 \;\; = \;\; \cos z\,\partial_u - \frac{F_u}{F}\,\sin z\,\partial_z \,, \tag{4.48}$$

$$K_7 \;\; = \;\; \partial_z \,. \tag{4.49}$$

The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = \mu\,K_4 \,, \qquad [K_2, K_4] = -\mu\,K_3 \,, \qquad [K_3, K_4] = \mu\,K_2 \,, \tag{4.50}$$

$$[K_5, K_6] = -\,k\,K_7 \,, \qquad [K_5, K_7] = -K_6 \,, \qquad [K_6, K_7] = K_5 \,. \tag{4.51}$$

Thus, the corresponding algebra for this case is $\mathcal{L}_{III} = \mathcal{L}_k \oplus \tau \oplus so\,(3)$. Here τ is associated to the Killing vector field K_1, whereas to the sub-algebra $so\,(3)$ correspond K_2, K_3 and K_4. Again $\mathcal{L}_k$ is generated by K_5, K_6 and K_7.

Class IV : $m^2 \neq 0$, any real k, $\omega = 0$. In the integration of the Killing equation for this general class one is led to distinguish two different subclasses according to $k \neq 0$ or $k = 0$. We shall denote these subclasses as classes IVa and IVb, respectively.

Class IVa : $m^2 \neq 0$, $k \neq 0$, $\omega = 0$. This class corresponds to the so-called degenerated Gödel-type manifolds. One obtains for this class the following Killing vector fields:

$$K_1 \;\; = \;\; \partial_t \,, \qquad K_2 = \partial_\phi \,, \tag{4.52}$$

$$K_3 \;\; = \;\; \cos\phi\,\partial_r - \frac{D_r}{D}\,\sin\phi\,\partial_\phi \,, \tag{4.53}$$

$$K_4 \;\; = \;\; -\sin\phi\,\partial_r - \frac{D_r}{D}\,\cos\phi\,\partial_\phi \,, \tag{4.54}$$

$$K_5 \;\; = \;\; \sin z\,\partial_u + \frac{F_u}{F}\,\cos z\,\partial_z \,, \tag{4.55}$$

$$K_6 \;\; = \;\; \cos z\,\partial_u - \frac{F_u}{F}\,\sin z\,\partial_z \,, \tag{4.56}$$

$$K_7 \;\; = \;\; \partial_z \,, \tag{4.57}$$

where $D(r) = (1/m) \sinh mr$ for $m^2 > 0$, or $D(r) = (1/\mu) \sin \mu r$ for $m^2 \equiv -\mu^2 < 0$, and the function $F(u)$ for $k \neq 0$ is given by (3.19). The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = K_4, \qquad [K_2, K_4] = -K_3, \qquad [K_3, K_4] = -m^2 K_2, \qquad (4.58)$$
$$[K_5, K_6] = -k K_7, \qquad [K_5, K_7] = -K_6, \qquad [K_6, K_7] = K_5, \qquad (4.59)$$

where one should substitute $-m^2$ by μ^2 if $m^2 < 0$. So, the corresponding Lie algebra is $\mathcal{L}_{IVa} = \mathcal{L}_k \oplus \tau \oplus \mathcal{L}_m$, where $\mathcal{L}_m$ is $so(2,1)$ for $m^2 > 0$, and $so(3)$ for $m^2 = -\mu^2 < 0$. The sub-algebra $\mathcal{L}_k$ (generated by K_5, K_6 and K_7) is $so(3)$ for $k < 0$, and $so(2,1)$ for $k > 0$. Again τ is associated to the Killing vector field K_1.

Class IVb : $m^2 \neq 0$, $k = 0$, $\omega = 0$. We shall refer to this class as doubly-degenerated Gödel-type manifolds. One obtains for this class the following Killing vector fields:

$$K_1 = \partial_t, \qquad K_2 = \partial_\phi, \qquad (4.60)$$

$$K_3 = \cos\phi \, \partial_r - \frac{D_r}{D} \sin\phi \, \partial_\phi, \qquad (4.61)$$

$$K_4 = -\sin\phi \, \partial_r - \frac{D_r}{D} \cos\phi \, \partial_\phi, \qquad (4.62)$$

$$K_5 = \sin z \, \partial_u + \frac{1}{u} \cos z \, \partial_z, \qquad (4.63)$$

$$K_6 = \cos z \, \partial_u - \frac{1}{u} \sin z \, \partial_z, \qquad (4.64)$$

$$K_7 = \partial_z, \qquad (4.65)$$

$$K_8 = u \sin z \, \partial_t + t \sin z \, \partial_u + \frac{1}{u} t \cos z \, \partial_z, \qquad (4.66)$$

$$K_9 = u \cos z \, \partial_t + t \cos z \, \partial_u - \frac{1}{u} t \sin z \, \partial_z, \qquad (4.67)$$

where again $D(r) = (1/m) \sinh mr$ for $m^2 > 0$, or $D(r) = (1/\mu) \sin \mu r$ for $m^2 \equiv -\mu^2 < 0$. The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = K_4, \qquad [K_2, K_4] = -K_3, \qquad [K_3, K_4] = -m^2 K_2, \qquad (4.68)$$
$$[K_5, K_7] = -K_6, \qquad [K_6, K_7] = K_5, \qquad [K_1, K_8] = K_5, \qquad (4.69)$$
$$[K_1, K_9] = K_6, \qquad [K_5, K_8] = K_1, \qquad [K_6, K_9] = K_1, \qquad (4.70)$$
$$[K_7, K_8] = K_9, \qquad [K_7, K_9] = -K_8, \qquad [K_8, K_9] = -K_7, \qquad (4.71)$$

where one should substitute $-m^2$ by μ^2 if $m^2 < 0$. So, the corresponding Lie algebra is $\mathcal{L}_{IVb} = t^3 \Subset so(2,1) \oplus \mathcal{L}_m$, where $\mathcal{L}_m$ is generated by K_2, K_3, K_4, and is either $so(2,1)$ or $so(3)$ depending on whether $m^2 > 0$ or $m^2 = -\mu^2 < 0$. The sub-algebra $t^3 \Subset so(2,1)$ is generated by $K_1, K_5, K_6, K_7, K_8, K_9$.

Class V : $m^2 = 0$, $k \neq 0$, $\omega = 0$. A set of linearly independent Killing vector field for this class is

$$K_1 = \partial_t, \qquad K_2 = \partial_\phi, \qquad (4.72)$$

$$K_3 = \cos\phi \, \partial_r - \frac{1}{r} \sin\phi \, \partial_\phi, \qquad (4.73)$$

$$K_4 = -\sin\phi \, \partial_r - \frac{1}{r} \cos\phi \, \partial_\phi, \qquad (4.74)$$

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$$K_5 = \sin z \; \partial_u + \frac{F_u}{F} \cos z \; \partial_z \,, \tag{4.75}$$

$$K_6 = \cos z \; \partial_u - \frac{F_u}{F} \sin z \; \partial_z \,, \tag{4.76}$$

$$K_7 = \partial_z \,, \tag{4.77}$$

$$K_8 = r \sin\phi \; \partial_t + t \sin\phi \; \partial_r + \frac{1}{r} \, t \cos\phi \; \partial_\phi \,, \tag{4.78}$$

$$K_9 = r \cos\phi \; \partial_t + t \cos\phi \; \partial_r - \frac{1}{r} \, t \sin\phi \; \partial_\phi \,, \tag{4.79}$$

where $F(u)$ depends upon the sign of k and is given by eq. (3.19).

The Lie algebra has the following nonvanishing commutators:

$$[K_2, K_3] = K_4 \,, \qquad [K_2, K_4] = -K_3 \,, \qquad [K_5, K_6] = -\,k\,K_7 \,, \tag{4.80}$$
$$[K_5, K_7] = -K_6 \,, \qquad [K_6, K_7] = K_5 \,, \qquad [K_1, K_8] = -K_4 \,, \tag{4.81}$$
$$[K_1, K_9] = K_3 \,, \qquad [K_4, K_8] = -K_1 \,, \qquad [K_3, K_9] = K_1 \,, \tag{4.82}$$
$$[K_2, K_8] = K_9 \,, \qquad [K_2, K_9] = -K_8 \,, \qquad [K_8, K_9] = -K_2 \,. \tag{4.83}$$

So, the corresponding Lie algebra is $\mathcal{L}_V = t^3 \in so\,(2,1) \oplus \mathcal{L}_k$, where $\mathcal{L}_k$ is generated by K_5, K_6, K_7, and is either $so\,(2,1)$ or $so\,(3)$ depending on whether $k > 0$ or $k < 0$. The sub-algebra $t^3 \in so\,(2,1)$ is generated by $K_1, K_2, K_3, K_4, K_8, K_9$.

Class VI : $m^2 = 0$, $k = 0$, $\omega = 0$. From (3.12) – (3.14) this case corresponds to the 5D flat manifold whose Lie algebra is $\mathcal{L}_{VI} = so\,(4,1)$ since it clearly has the well known fifteen Killing vector fields, namely five translations, four spacetime rotations, and six space rotations.

It is worth noting that none of the above Lie algebras is semi-simple, but some of their sub-algebras are. Besides, most of the simple sub-algebras are noncompact. The 3D sub-algebra $so\,(3)$ present in all classes is compact, though.

The number of Killing vector fields we have found for each of the above six classes makes explicit that the 5D locally homogeneous generalized Gödel-type manifolds admit a group of isometry G_7 when ($1a$): $m^2 \neq 4\,\omega^2$, any real k, $\omega \neq 0$, or when ($1b$): $m^2 \neq 0$, $k \neq 0$, $\omega = 0$. Groups G_9 of isometry occur when ($2a$): $m^2 = 4\,\omega^2$, any real k, $\omega \neq 0$, or ($2b$): $m^2 \neq 0$, $k = 0$, $\omega = 0$, or when ($2c$): $m^2 = 0$, $k \neq 0$, $\omega = 0$. Clearly when $m^2 = \omega = k = 0$ there is G_{15}. These possible groups are in agreement with theorem 3 of the previous section. Actually the integration of the Killing equations constitutes a different way of deriving that theorem. Furthermore, these equations also show that the isotropy subgroup H of G_r is such that $\dim(H) = 2$ when the above conditions ($1a$) and ($1b$) are satisfied, while the conditions ($2a$), ($2b$) and ($2c$) lead to $\dim(H) = 4$, also in agreement with the previous section. Clearly $\dim(H) = 10$ when $m^2 = \omega = k = 0$.

5 Causal Anomalies and Final Remarks

In this section we shall initially be concerned with the problem of causal anomalies in the generalized Gödel-type manifolds. Then we proceed by examining whether the IM gravity allows solutions of generalized Gödel-type metrics (3.16). Finally, we conclude by addressing to the general question as to whether the IM gravity theory rules out the 4D noncausal Gödel-type solutions to Einstein's equations of general relativity.

In the first three of the six classes of homogeneous generalized Gödel-type manifolds we have discussed in Section 3, there are closed timelike curves. Indeed, the analysis made in a previous paper [64] can be easily extended to the generalized 5D Gödel-type manifolds of the present article. To this end, we write the line element (3.16) in the form

$$ds^2 = dt^2 + 2\,H(r)\,dt\,d\phi - dr^2 - G(r)\,d\phi^2 - F^2(u)\,dz^2 - du^2\,, \tag{5.1}$$

where $G(r) = D^2 - H^2$ and (r, ϕ, z) are cylindrical coordinates. Now, the existence of closed timelike curves of the Gödel-type depends on the behavior of $G(r)$. Indeed, if $G(r) < 0$ for a certain range of r ($r_1 < r < r_2$, say), Gödel's circles [74] $u, t, z, r = const$ are closed timelike curves.

Since one can always make $H = 0$ for the generalized Gödel-type manifolds of classes IV, V and VI, then $G(r) > 0$ for all $r > 0$. Thus there are no closed timelike Gödel's circles in these classes of manifolds.

On the other hand, following the above-outlined reasoning it easy to show (see [64] for details) that for each of the remaining three classes (Class I to Class III) one can always find a critical radius r_c such that for all $r > r_c$ one has $G(r) < 0$, making clear that there are closed timelike curves in these families of homogeneuous generalized Gödel-type manifolds. However, in what follows we shall show that these types of noncausal *curved* manifolds are not permitted in the context of the induced matter theory.

In the Lorentz frame $\widehat{\Theta}^A$ given by (4.1) the nonvanishing frame components of the Einstein tensor $\widehat{G}_{AB} = \widehat{R}_{AB} - \frac{1}{2}\,R\,\widehat{\eta}_{AB}$ are

$$\widehat{G}_{00} = -\frac{D''}{D} + \frac{3}{4}\left(\frac{H'}{D}\right)^2 - \frac{\ddot{F}}{F}\,, \tag{5.2}$$

$$\widehat{G}_{02} = \frac{1}{2}\left(\frac{H'}{D}\right)'\,, \tag{5.3}$$

$$\widehat{G}_{11} = \widehat{G}_{22} = \frac{1}{4}\left(\frac{H'}{D}\right)^2 + \frac{\ddot{F}}{F}\,, \tag{5.4}$$

$$\widehat{G}_{33} = \widehat{G}_{44} = \frac{D''}{D} - \frac{1}{4}\left(\frac{H'}{D}\right)^2\,, \tag{5.5}$$

where the prime and dot denote derivative with respect to r and u, respectively.

The field equations (1.2) require that $\widehat{G}_{02} = 0$, which in turn implies that

$$\frac{H'}{D} = \text{const} \equiv -2\,\omega\,. \tag{5.6}$$

Inserting (5.6) into (5.4), (5.5) and (5.2) one easily finds that the IM field equations are fulfilled if and only if the independent parameters ω, k and m^2 [see eqs. (3.9) − (3.10] vanish identically, which leads to the only solution given by

$$H = a\,, \qquad D = b\,r + c\,, \qquad \text{and} \qquad F = \beta\,u + \gamma\,, \tag{5.7}$$

where a, b, c, β, and γ are arbitrary real constants. However, these constants have no physical meaning, and can be taken to be $a = c = \gamma = 0$ and $b = \beta = 1$ by a suitable choice of coordinates. Indeed, if one performs the coordinate transformations

$$t = \bar{t} - \frac{a}{b}\,\bar{\phi}\,, \qquad\qquad r = \bar{r} - \frac{c}{b}\,, \tag{5.8}$$

$$\phi = \frac{\bar{\phi}}{b}\,, \qquad z = \frac{\bar{z}}{\beta}\,, \qquad u = \bar{u} - \frac{\gamma}{\beta}\,, \tag{5.9}$$

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the line element (5.1) becomes

$$d\hat{s}^2 = d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2 \, d\bar{\phi}^2 - d\bar{z}^2 - d\bar{u}^2 \, , \qquad (5.10)$$

in which we obviously have $G(\bar{r}) = \bar{r}^2 > 0$ for $\bar{r} \neq 0$. The line element (5.10) corresponds to a manifestly flat 5D manifold, making it clear that the underlying manifold can be taken to be the simply connected Euclidean manifold $\mathbb{R}^5$, and therefore as $G(\bar{r}) > 0$ no closed timelike circles are permitted. Furthermore the above results clearly show that the IM theory does not admit any *curved* 5D Gödel-type metric (3.16) as solution to its field equations (1.2).

However, in a recent work Mc Manus [17] has shown that a one-parameter family of solutions of the field equations (1.2) previously found by Ponce de Leon [75] was in fact flat in five dimensions. And yet the corresponding 4D induced models were shown to be a perfect fluid family of Friedmann-Robertson-Walker curved models (see Refs. [11, 13, 19] and also [76] – [78], where other Riemann-flat solutions are also discussed).

Therefore a question which naturally arises here is whether the above 5D flat metric, which is the only solution to the IM field equations, can similarly give rise to any 4D *curved* spacetime. However, from (5.10) one obviously has that the corresponding 4D spacetime is nothing but the Minkowski flat space (this result can also be derived by using a computer algebra package as, e.g., CLASSI [71, 69] to calculate the 4D curvature tensor for $m^2 = \omega = 0$). In brief, the only solution of the IM field equations (1.2) of generalized Gödel-type is the 5D flat space (5.7), which give rise only to the 4D Minkowski (flat) spacetime, whose topology can be taken to be the simply connected Euclidean $\mathbb{R}^5$, in which no closed timelike curves are permitted.

Although the above results can be looked upon as if the induced matter theory works as an effective therapy for the causal anomalies which arises when one starts from the specific generalized 5D Gödel-type family of metrics (5.1), this does not ensure that the induced matter version of general relativity is an efficient treatment for the causal anomalies (solutions with closed timelike curves) in general relativity as it has been conjectured in [65]. Actually, in a recent paper (which unfortunately has not been initially noticed by Rebouças and Teixeira [65]) Romero *et al.* [67] (see also [79]) have shown that the induced matter 5D scheme is indeed general enough to locally generate all solutions to 4D Einstein's field equations. This is ensured by a theorem due to Campbell [80] which states that any analytic n-dimensional Riemannian space can be locally embedded in a $(n + 1)$-dimensional Ricci-flat space. In our context this amounts to saying that there must exist a five-dimensional Ricci-flat space which locally gives rise to the 4D Gödel noncausal solution of Einstein's equations of general relativity. Thus, what still remains to be done regarding Gödel-type spaces is to find out this 5D Ricci-flat space which gives rise (locally) to the 4D Gödel-type spacetimes of general relativity.

To conclude it is worth stressing some features of the local underlying embedding of the induced matter theory. Any Riemann-flat manifold obviously is also Ricci-flat. The reverse, however, does not necessarily holds, and one can have Ricci-flat spaces which are not Riemann-flat. For the generalized 5D Gödel-type geometries we have discussed in this paper the condition for Ricci-flatness ($\widehat{R}_{AB} = 0$) necessarily leads to Riemann-flat spaces. Remarkably many solutions of the field equations (1.2) are indeed Riemann-flat (see [11, 17, 19] and [75] – [78]). From a purely mathematical 5D point of view all Riemann-flat spaces are locally equivalent (locally isometric). However, from the viewpoint of the 5D induced matter gravity all the above-referred 5D Riemann-flat solutions give rise to physically (and geometrically) distinct 4D spacetimes [11, 17, 19], [75] – [78]. On the other

hand, in the light of the equivalence problem techniques we have discussed in Section 2, these 5D Riemann-flat examples also show that all 5D Cartan scalars (2.3) can vanish identically, with or without the vanishing of the corresponding (induced) 4D Cartan scalars.

Acknowledgement

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